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Strong homology does not have compact supports

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Abstract

The paper exhibits examples of paracompact spaces X with nonvanishing strong homology groups, while the (strong) homology groups with compact supports vanish. Moreover, all the Čech homology groups of X vanish.

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1. Introduction

Strong homology groups $\overline{H}_p(X,A;G)$, for pairs of topological spaces (X,A), were defined by Yu.T. Lisitsa and the author in 1983 [5] (also see ([6,9]). For pairs (X,A), where A is normally embedded in X, e.g., if X is paracompact and A is closed, these groups satisfy all the Eilenberg–Steenrod axioms [7,8,9,2]. Moreover, they are invariants of strong shape [7].

For compact metric spaces strong homology coincides with Steenrod homology [9] and for compact Hausdorff spaces, it coincides with homology theories developed by various authors (for a survey of results see [9]). In particular, it coincides with Massey's homology [17] (see [3]).

In the noncompact case, one also defines strong homology groups with compact supports $\overline{H}^c_p(X,A;G)$ as direct limits of strong groups $\overline{H}_p(K,K\cap A;G)$, where K ranges over all compact subsets $K\subseteq X$, ordered by inclusion $\subseteq$. The homomorphisms $\overline{H}_p(K,K\cap A;G)\to\overline{H}_p(X,A;G)$, induced by inclusion, define a homomorphism

$$\overline{H}^c_p(X,A;G)\to\overline{H}_p(X,A;G). \tag{1}$$

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It is natural to ask if this homomorphism is always an isomorphism, i.e., does strong homology have compact supports?

In [14] a p-dimensional separable metric space X was exhibited, $p \geqslant 0$, which has the property that $\overline{H}^c_{p-1}(X;\mathbb{Z}) = 0$ and $\overline{H}_{p-1}(X;\mathbb{Z}) = \lim^1 \boldsymbol{A}$, where $\boldsymbol{A}$ was a certain Abelian progroup. The continuum hypothesis implied that $\lim^1 \boldsymbol{A} \neq 0$, giving thus a negative answer to the above question. Subsequently, Dow, Simon and Vaughan [1] showed that the question, whether $\lim^1 \boldsymbol{A} = 0$ or $\lim^1 \boldsymbol{A} \neq 0$, is ZFC-undecidable. Consequently, the question whether the example from [14] shows that the two theories differ, is also ZFC-undecidable.

Recently, Günther showed, without using the ZFC-axioms, that strong homology does not have compact supports [3]. The example which enabled him to draw this conclusion is the pair (X, A), where X is the set of all countable ordinals and A is the subset of the limit ordinals of X. This subset is closed and normally embedded in X and

$$\overline{H}^c_0(X, A;\mathbb{Z}/2) \to \overline{H}_0(X, A;\mathbb{Z}/2) \tag{2}$$

is not an isomorphism. Notice that in Günther's example X is not paracompact. Moreover, $\dim X = 0$ and therefore, the strong homology groups $\overline{H}_0(X, A;\mathbb{Z}/2)$ coincide with the Čech groups [16].

The purpose of the present paper is to produce (also using only the ZFC-axioms) more sophisticated examples, which improve Günther's example in several respects. In particular, we will establish the following theorem.

Theorem 1. *There exists a paracompact space X, all of whose strong homology groups with compact supports and all Čech groups vanish,*

$$\overline{H}^c_p(X;G) = 0, \quad p \in \mathbb{Z}, \tag{3}$$

$$\check{H}_p(X;G) = 0, \quad p \geqslant 0. \tag{4}$$

Nevertheless, in dimensions $p \geqslant -1$, all the strong homology groups with integer coefficients are nontrivial,

$$\overline{H}_p(X;\mathbb{Z}) \neq 0, \quad p \geqslant -1. \tag{5}$$

The construction uses in an essential way a collection of spaces, recently defined by the author and denoted by $X(m, n, \Lambda)$ [12].

2. The spaces $X(m, n, \Lambda)$

Let $m \geqslant 1$, $n \geqslant 1$, be integers and let $(\Lambda, \leqslant)$ be an infinite directed set. In [12] an inverse system $\boldsymbol{X} = \boldsymbol{X}(m, n, \Lambda)$ of CW-complexes $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ was defined as follows. Let $(B^m, *)$ denote the standard m-cell with a base-point $*$ chosen on its boundary $\partial B^m = S^{m-1}$. For each $\lambda \in \Lambda$ and each increasing sequence of indices $\lambda_0 \leqslant \cdots \leqslant \lambda_n$ from Λ, let

$$X^{\lambda}_{\lambda_0\ldots\lambda_n} = \begin{cases} S^m, & \lambda \leqslant \lambda_0, \\ B^m, & \lambda \nleqslant \lambda_0. \end{cases} \tag{1}$$

Let X_λ be the wedge of the collection of all $X^{\lambda}_{\lambda_0\ldots\lambda_n}$, $\lambda_0 \leqslant \cdots \leqslant \lambda_n$ (weak topology),

$$X_\lambda = \bigvee_{\lambda_0 \leqslant \cdots \leqslant \lambda_n} X^{\lambda}_{\lambda_0\ldots\lambda_n}. \tag{2}$$

Furthermore, for $\lambda \leqslant \lambda'$, let $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ be the wedge

$$p_{\lambda\lambda'} = \bigvee_{\lambda_0 \leqslant \cdots \leqslant \lambda_n} p^{\lambda\lambda'}_{\lambda_0\ldots\lambda_n} \tag{3}$$

of the mappings $p^{\lambda\lambda'}_{\lambda_0\ldots\lambda_n} : X^{\lambda'}_{\lambda_0\ldots\lambda_n} \to X^{\lambda}_{\lambda_0\ldots\lambda_n}$, defined as follows. If $\lambda' \leqslant \lambda_0$ or if $\lambda' \nleqslant \lambda_0$ and $\lambda \nleqslant \lambda_0$, then $p^{\lambda\lambda'}_{\lambda_0\ldots\lambda_n}$ is the identity mapping on S^m and B^m respectively. In the only remaining case, $\lambda' \nleqslant \lambda_0$ and $\lambda \leqslant \lambda_0$, one has $X^{\lambda'}_{\lambda_0\ldots\lambda_n} = B^m$, but $X^{\lambda}_{\lambda_0\ldots\lambda_n} = S^m$. In this case, let $p^{\lambda\lambda'}_{\lambda_0\ldots\lambda_n} = \phi$, where $\phi : B^m \to S^m$ is the mapping, which collapses the boundary of B^m to the base-point $*$ of S^m.

Moreover, a space $X = X(m, n, \Lambda)$ and mappings $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, were defined as follows. As a set, X is given by the wedge

$$X = \bigvee_{\lambda_0 \leqslant \cdots \leqslant \lambda_n} X_{\lambda_0\ldots\lambda_n}, \tag{4}$$

where $X_{\lambda_0\ldots\lambda_n} = B^m$, for all increasing sequences $\lambda_0 \leqslant \cdots \leqslant \lambda_n$ in Λ.

$$p_\lambda = \bigvee_{\lambda_0 \leqslant \cdots \leqslant \lambda_n} p^{\lambda}_{\lambda_0\ldots\lambda_n}, \tag{5}$$

where $p^{\lambda}_{\lambda_0\ldots\lambda_n} : X_{\lambda_0\ldots\lambda_n} \to X^{\lambda}_{\lambda_0\ldots\lambda_n}$ is given by

$$p^{\lambda}_{\lambda_0\ldots\lambda_n} = \begin{cases} \phi, & \lambda \leqslant \lambda_0, \\ \mathrm{id}, & \lambda \nleqslant \lambda_0. \end{cases} \tag{6}$$

X is topologized by taking for a basis of the topology the collection of all sets $(p_\lambda)^{-1}(V_\lambda)$, where $\lambda \in \Lambda$ and V_λ is an open subset of X_λ.

We now quote some results from [12] (see Theorems 6 and 4).

Lemma 1. $\boldsymbol{X} = \boldsymbol{X}(m, n, \Lambda)$ *is an inverse system of CW-complexes. The space* $X = X(m, n, \Lambda)$ *is paracompact and the mappings* $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, *define a resolution* $\boldsymbol{p} : X \to \boldsymbol{X}$ *in the sense of* [10] (*also see* [15]). *Consequently, X is the inverse limit of* $\boldsymbol{X}$. *Moreover, the* leaves $X_{\lambda_0\ldots\lambda_n}$ *are closed in X and inherit from X the usual topology of* B^m.

Lemma 2. *If $\boldsymbol{X} = \boldsymbol{X}(m,n,\Lambda)$ and the cofinality* $\mathrm{cof}(\Lambda) = \aleph_{n-1}$, *then the nth derived limit of the mth homology with integer coefficients*

$$\lim{}^n H_m(\boldsymbol{X};\mathbb{Z}) \neq 0. \tag{7}$$

3. Homology with compact supports of $X(m,n,\Lambda)$

Theorem 2. *Let Λ be a cofinite uncountable directed set, e.g., the set of all finite subsets of an uncountable set. Then all strong homology groups with compact supports of the space $X = X(m,n,\Lambda)$ vanish,*

$$\overline{H}{}^{c}_m(X;G) = 0. \tag{1}$$

Proof. The union of a finite collection of leaves $X_{\lambda_0\ldots\lambda_n}$ of X is homeomorphic to the wedge $B^m \vee \cdots \vee B^m$, which is a contractible space. Therefore, its strong groups vanish. Consequently, the assertion of Theorem 2 is an immediate consequence of the following lemma.

Lemma 3. *Let Λ and X be as in Theorem 2. Then every compact subset $K \subseteq X$ is contained in the union of a finite collection of leaves of X.*

Proof. Assume the contrary, i.e., that there is a sequence of different n-tuples $(\lambda_0^i,\ldots,\lambda_n^i)$ and there is a sequence of points $x^i \in K \cap (X_{\lambda_0^i\ldots\lambda_n^i}\backslash\{*\})$, $i = 1,2,\ldots.$ Since Λ is cofinite, for each i, the set M^i of all predecessors of λ_0^i is finite. Therefore, $M = \bigcup_i M^i$ is also a countable set. Since $|\Lambda| > \aleph_0$, there exists a $\lambda \in \Lambda\backslash M$. Clearly, for each i, $\lambda \notin M^i$, i.e., $\lambda \not\leqslant \lambda_0^i$. Consequently, by 2(1),

$$\left(X^{\lambda}_{\lambda_0^i\ldots\lambda_n^i},*\right) = (B^m,*) = \left(X_{\lambda_0^i\ldots\lambda_n^i},*\right)$$

and, by 2(6), $p^{\lambda}_{\lambda_0^i\ldots\lambda_n^i} = \mathrm{id}$, for all i. Therefore, the points $p_\lambda(x^i) = p^{\lambda}_{\lambda_0^i\ldots\lambda_n^i}(x^i) = x^i$ belong to a sequence of different leaves $X^{\lambda}_{\lambda_0^i\ldots\lambda_n^i}\backslash\{*\}$ of X_λ. This is a contradiction, because all the points $p_\lambda(x^i)$ belong to the compact set $p_\lambda(K)$, which must be contained in a finite subcomplex of X_λ, hence, in the union of a finite collection of leaves of X_λ.

Remark 1. The same argument proves that, for any homology theory, the homology groups with compact supports of X vanish. In particular, the singular homology groups and the Čech homology groups with compact supports vanish.

4. Čech groups of $X(m,n,\Lambda)$

The Čech homology groups $\check{H}_p(X;G)$ of a space X can be defined as the inverse limits of the inverse systems of Abelian groups $H_p(\boldsymbol{X};G)$, where $\boldsymbol{p}: X \to \boldsymbol{X}$ is an HPol-expansion of X (see [15, Chapter II, § 3.1]). In particular, one can use as $\boldsymbol{p}$ any resolution of X consisting of CW-complexes (see [15, Chapter I, § 6, Theorem 2]).

Theorem 3. *For* $X = X(m,n,\Lambda)$, $\boldsymbol{X} = \boldsymbol{X}(m,n,\Lambda)$ *and any* p,

$$\check{H}_p(X;G) = 0. \tag{1}$$

Proof. First note that Lemma 1 implies

$$\check{H}_p(X;G) = \lim H_p(\boldsymbol{X};G). \tag{2}$$

Next note that, by 2(2),

$$H_p(X_\lambda;G) = \bigoplus_{\lambda_0\leqslant\cdots\leqslant\lambda_n} H_p\big((X^\lambda_{\lambda_0\ldots\lambda_n},*);G\big). \tag{3}$$

Since $H_p((S^m,*);G) = H_p((B^m,*);G) = 0$, for $p \neq m$, one concludes that

$$H_p(X_\lambda;G) = 0, \quad p \neq m, \tag{4}$$

and thus also $H_p(\boldsymbol{X};G) = \boldsymbol{0}$. Hence, (1) holds, for $p \neq m$.

For $p = m$, note that $H_m((S^m,*);G) = G$, $H_m((B^m,*);G) = 0$ and therefore, (3) implies

$$H_m(X_\lambda;G) = \bigoplus_{\lambda\leqslant\lambda_0\leqslant\cdots\leqslant\lambda_n} G. \tag{5}$$

Moreover, by 2(3), $p_{\lambda\lambda'\#}\colon H_m(X_{\lambda'};G) \to H_m(X_\lambda;G)$, $\lambda \leqslant \lambda'$, is the natural inclusion. Let $\alpha = (\alpha_\lambda) \in \lim H_m(X_\lambda;G)$ and let $\alpha_\lambda \in H_m(X_\lambda;G)$ be contained in the subgroup of $H_m(X_\lambda;G)$, obtained by summing up copies of G over a finite collection of sequences $\lambda_0^i \leqslant \cdots \leqslant \lambda_n^i$, $i = 1,\ldots,k$. Choose $\lambda' \in \Lambda$ so that $\lambda' > \lambda$, $\lambda_0^1,\ldots,\lambda_0^k$. Then the corresponding summands do not appear in $H_m(X_{\lambda'};G)$. Therefore, $\alpha_\lambda = p_{\lambda\lambda'}(\alpha_{\lambda'}) = 0$, which shows that $\alpha = 0$, i.e., (1) holds also for $p = m$.

5. Strong groups of $X(m,2,\Lambda)$

Originally, the strong homology group $\overline{H}_p(X;G)$ of a space X was defined as the strong homology group $\overline{H}_p(\boldsymbol{X};G)$ of a cofinite inverse system X of ANR's such that X admits a resolution $\boldsymbol{p}\colon X \to \boldsymbol{X}$. It was later shown that one can also use resolutions consisting of CW-complexes (see [13, § 3] or [11, Theorem 5]). Therefore, for $X = X(m,n,\Lambda)$ and $\boldsymbol{X} = \boldsymbol{X}(m,n,\Lambda)$ one has

$$\overline{H}_p(X;G) = \overline{H}_p(\boldsymbol{X};G). \tag{1}$$

Strong homology groups of an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ are defined as homology groups of the total complex associated with a certain double chain complex. One defines also a sequence of groups $\check{H}^r_p(\boldsymbol{X};G)$, $r \geqslant 0$, called the *Čech groups of height* r, beginning with the usual Čech group

$$\check{H}^0_p(\boldsymbol{X};G) \approx \lim H_p(\boldsymbol{X};G). \tag{2}$$

These groups are related by exact sequences, discovered in 1984 by Miminoshvili [18] (for a proof see [14,13] or [19]).

$$\begin{aligned} \cdots \to \check{H}^{r-2}_{p+1}(\boldsymbol{X};G) \to \lim{}^r H_{p+r}(\boldsymbol{X};G) \to \check{H}^r_p(\boldsymbol{X};G) \\ \to \check{H}^{r-1}_p(\boldsymbol{X};G) \to \lim{}^{r+1} H_{p+r}(\boldsymbol{X};G) \to \cdots, \end{aligned} \tag{3}$$

$$0 \to \lim{}^1 \check{H}^*_{p+1}(\boldsymbol{X};G) \to \overline{H}_p(\boldsymbol{X};G) \to \lim \check{H}^*_p(\boldsymbol{X};G) \to 0, \tag{4}$$

where $\check{H}^*_p(\boldsymbol{X};G)$ denotes the inverse sequence

$$\check{H}^*_p(\boldsymbol{X};G) = \check{H}^0_p(\boldsymbol{X};G) \leftarrow \cdots \leftarrow \check{H}^{r-1}_p(\boldsymbol{X};G) \leftarrow \check{H}^r_p(\boldsymbol{X};G) \leftarrow \cdots. \tag{5}$$

Lemma 4. *If X is an inverse system of CW-complexes of bounded dimension, then, for all p,*

$$\overline{H}_p(\boldsymbol{X};G) \approx \lim \check{H}^*_p(\boldsymbol{X};G). \tag{6}$$

Proof. For any given p, and r sufficiently large, $p+1+r$ exceeds the dimension of all X_λ, $\lambda \in \Lambda$. Therefore, $H_{p+1+r}(X_\lambda;G) = 0$, $\lambda \in \Lambda$, which implies $H_{p+1+r}(\boldsymbol{X};G) = 0$, and a fortiori,

$$\lim{}^{r+1} H_{p+1+r}(\boldsymbol{X};G) = 0. \tag{7}$$

It now follows from (3) that

$$\check{H}^r_{p+1}(\boldsymbol{X};G) \to \check{H}^{r-1}_{p+1}(\boldsymbol{X};G) \tag{8}$$

is an epimorphism, for all sufficiently large r. Consequently, the sequence $\check{H}^*_{p+1}(\boldsymbol{X};G)$ has the Mittag-Leffler property. However, it is well known that the first derived limit of such a sequence vanishes (see, e.g., [15, Chapter II, § 6.2, Theorem 10]). Therefore, (4) yields the desired conclusion (6).

Lemma 5. *If $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inverse system of spaces indexed by a directed set Λ of cofinality $\operatorname{cof}(\Lambda) = \aleph_{n-1}$, $n \geqslant 1$, then, for any p and any $r \geqslant n$, there exists an epimorphism*

$$\lim \check{H}^*_p(\boldsymbol{X};G) \to \check{H}^{r-1}_p(\boldsymbol{X};G). \tag{9}$$

Proof. It is well known that $\lim^s \boldsymbol{H} = 0$, for any inverse system of Abelian groups $\boldsymbol{H}$, indexed by Λ, and any $s \geqslant n+1$ (see e.g., [4, Corollary 3.2]). Therefore, by (3),

$$\check{H}^r_p(\boldsymbol{X};G) \to \check{H}^{r-1}_p(\boldsymbol{X};G) \tag{10}$$

is an epimorphism, for any $r \geqslant n$. Consequently, all bonding homomorphisms in the sequence

$$\check{H}^{n-1}_p(\boldsymbol{X};G) \leftarrow \check{H}^n_p(\boldsymbol{X};G) \leftarrow \cdots \tag{11}$$

are epimorphisms and thus, the projections of the limit group $\lim \check{H}^*_p(\boldsymbol{X};G)$ to any of the terms of (11) is also an epimorphism.

Theorem 4. *Let* $X = X(m, 2, \Lambda)$ *and* $\boldsymbol{X} = \boldsymbol{X}(m, 2, \Lambda)$, *where* $\operatorname{cof}(\Lambda) = \aleph_1$, $m \geqslant 1$. *Then the strong groups*

$$\overline{H}_{m-2}(\boldsymbol{X}; \mathbb{Z}) \neq 0, \qquad \overline{H}_{m-2}(X; \mathbb{Z}) \neq 0. \tag{12}$$

Proof. By Lemma 2,

$$\lim{}^2 H_m(\boldsymbol{X}; \mathbb{Z}) \neq 0. \tag{13}$$

By (2) and Theorem 3,

$$\check{H}^0_{m-1}(\boldsymbol{X}; \mathbb{Z}) \approx \lim H_{m-1}(\boldsymbol{X}; \mathbb{Z}) = 0. \tag{14}$$

Therefore, for $r = 2$ and $p = m - 2$, (3) yields the exact sequence

$$0 \to \lim{}^2 H_m(\boldsymbol{X}; \mathbb{Z}) \to \check{H}^2_{m-2}(\boldsymbol{X}; \mathbb{Z}). \tag{15}$$

Hence, (13) implies

$$\check{H}^2_{m-2}(\boldsymbol{X}; \mathbb{Z}) \neq 0. \tag{16}$$

Applying Lemma 5, for $n = 2$, $p = m - 2$ and $r = 3$, one concludes that there is an epimorphism

$$\lim H^*_{m-2}(\boldsymbol{X}; \mathbb{Z}) \to \check{H}^2_{m-2}(\boldsymbol{X}; \mathbb{Z}). \tag{17}$$

Consequently, (16) implies

$$\lim H^*_{m-2}(\boldsymbol{X}; \mathbb{Z}) \neq 0. \tag{18}$$

Now (1) follows by applying Lemma 4.

6. Proof of the main theorems

Combining all previous theorems we now obtain our main technical result.

Theorem 5. *Let* $X = X(m, n, \Lambda)$, $m \geqslant 1$, *where* Λ *is a cofinite directed set of cofinality* $\operatorname{cof}(\Lambda) = \aleph_1$. *Then* X *is a paracompact space of dimension* $\dim X = m$. *All Čech homology groups and all strong homology groups with compact supports of* X *vanish. Nevertheless,*

$$\overline{H}_{m-2}(X; \mathbb{Z}) \neq 0. \tag{1}$$

Proof. It only remains to show that $\dim X = m$. Since X admits a resolution consisting of spaces X_λ of dimension $\dim X_\lambda \leqslant m$, it follows that $\dim X \leqslant m$ (see e.g., [15, Chapter I, § 6.2, Theorem 4]). On the other hand, X contains copies of B^m as closed subsets and therefore, $\dim X \geqslant m$.

Remark 2. That Čech homology does not have compact supports was shown in 1953 by an example due to Mishchenko [20].

Corollary 1. *The space* $X = X(1,2,\Lambda)$ *is paracompact,* $\dim X = 1$ *and*

$$\overline{H}_{-1}(X;\mathbb{Z}) \neq 0. \tag{2}$$

Remark 3. The phenomenon that strong homology groups in negative dimensions need not vanish was first demonstrated in [14]. However, the proof depended on the continuum hypothesis.

Proof of Theorem 1. Let Λ be a cofinite directed set with $\operatorname{cof}(\Lambda) = \aleph_1$. For each $m \geqslant 1$, put $X_m = X(m,2,\Lambda)$ and let X be the topological sum (coproduct)

$$X = \coprod_{m\geqslant 1} X_m. \tag{3}$$

Clearly, X is an infinite-dimensional paracompact space. Since $\check{H}_p(X_m;G) = 0$, for each m, and the Čech homology is additive (see [14, § 7]), the Čech groups of X also vanish.

In order to prove that the groups with compact supports vanish, notice that every compact subset $K \subseteq X$ is the finite sum of a collection of compact subsets $K_m \subseteq X_m$, $m = 1,\dots,k$. Since strong homology is additive with respect to finite sums, one concludes that

$$\overline{H}_p(K;G) = \bigoplus_{m=1}^{k} \overline{H}_p(K_m;G). \tag{4}$$

However, by Theorem 2, every class $\alpha_m \in \overline{H}_p(K_m;G)$ admits a compact set K'_m, such that $K_m \subseteq K'_m \subseteq X_m$ and the homomorphism $\overline{H}_p(K_m;G) \to \overline{H}_p(K'_m;G)$ annihilates α_m. Therefore, every class α from $\overline{H}_p(K;G)$ is annihilated by the homomorphism induced by $K \subseteq K'_1 \cup \dots \cup K'_m$.

Finally, since X_m is a retract of X, $\overline{H}_{m-2}(X_m;\mathbb{Z})$ is a direct summand of $\overline{H}_{m-2}(X;\mathbb{Z})$. Consequently, by Theorem 5, $\overline{H}_{m-2}(X;\mathbb{Z}) \neq 0$, for all $m \geqslant 1$.

Problem 1. Is there a separable metric space X, whose strong homology groups and strong homology groups with compact supports differ?

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